PMM U.S.S.R.,Vol.54,No.3,pp.331-337,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

DIFFERENTIAL GAMES WITH INTEGRAL CONSTRAINTS ON DISTURBANCES*

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A positional differential game for a linear system is considered. This game differs from games previously studied in the literature in that it does not assume geometrical constraints on the controls and the disturbances, but instead imposes integral constraints on the samples of the disturbance. The existence of an optimal strategy is established and a method of construction is suggested. Counterstrategies are constructed that generate the worst-case realizations of the disturbance. It is established that this differential game has a value and a saddle point.

1. Consider a differential game for a system whose motion is described by the equation

$$x^{\cdot} = A(t)x + B(t)u + C(t)v$$
$$u \in R^r, \quad v \in R^s, \quad t_0 \leqslant t \leqslant \vartheta \tag{1.1}$$

where x is the n-dimensional phase vector, u is the control vector, v is the disturbance vector, $A(t), B(t)$, and $C(t)$ are continuous matrix functions and t_0 and ϑ are fixed instants of time. Each possible realization of the disturbance $v[t_0[\cdot]\vartheta) = \{v[t], t_0 \leqslant t < \vartheta\}$ is Borel measurable and satisfies the constraint

$$I_v(t_0, \vartheta) \leqslant \nu[t_0], \quad I_v(\alpha, \beta) = \int_\alpha^\beta \langle v[t] \cdot \Psi(t) v[t] \rangle \, dt$$

Here $\nu[t_0] > 0$ is a given number, $\langle a \cdot b \rangle$ is the scalar product of the vectors a and b, $\Psi(t)$ is a matrix function continuous for $t_0 \leqslant t \leqslant \vartheta$, and $\langle v \cdot \Psi(t) v \rangle$ is a positive definite quadratic form. A strategy is any function

$$u(\cdot) = \{u(t, x, \nu, \varepsilon), \quad t_0 \leqslant t \leqslant \vartheta, \quad x \in R^n, \ 0 \leqslant \nu \leqslant \nu[t_0], \quad \varepsilon > 0\}$$

Assume that the strategy $u(\cdot)$ has been chosen, the position $\{t_*, x_*, \nu[t_*]\}$, $t_* \in [t_0, \vartheta)$, $0 \leqslant \nu[t_*] \leqslant \nu[t_0]$, is realized, the value $\varepsilon > 0$ has been chosen, and the partition $\Delta_u\{t_i\}$, $i = 1, \ldots, k+1$, has been fixed for the interval $[t_*, \vartheta]$, $t_1 = t_*, \ldots, t_{k+1} = \vartheta$. Then the motion $x[t_*[\cdot]\vartheta] = \{x[t], t_* \leqslant t \leqslant \vartheta\}$ is defined for $t_* \leqslant t \leqslant \vartheta$ as the solution of the stepwise differential equation

$$x^{\cdot}[t] = A(t)x[t] + B(t)u(t_i, x[t_i], \nu[t_i], \varepsilon) + C(t)v[t], \quad t_i \leqslant t < t_{i+1}, \quad i = 1, \ldots, k$$

where the realization $v[t_*[\cdot]\vartheta]$ may be any measurable function that satisfies the constraint

$$I_v(t_*, \vartheta) \leqslant \nu[t_*] \tag{1.2}$$

For any admissible realization $v[t_*[\cdot]\vartheta]$ (1.2), the function $\nu[t]$, $t_* \leqslant t \leqslant \vartheta$, that determines the remaining disturbance slack at time t, is defined by the equation

$$\nu[t] = \nu[t_*] - I_v(t_*, t), \quad t \in [t_*, \vartheta]$$

The function $\nu[t]$ can be determined during the control process by analysing the motion.

A counterstrategy is any function

$$v(\cdot) = \{v(t, x, \nu, \varepsilon), \quad t_0 \leqslant t \leqslant \vartheta, \quad x \in R^n, \ 0 \leqslant \nu \leqslant \nu[t_0], \ \varepsilon > 0\}$$

Assume that the counterstrategy $v(\cdot)$ has been chosen, the position $\{t_*, x_*, \nu[t_*]\}$ is realized, the value $\varepsilon > 0$ has been chosen, and the partition $\Delta_v\{t_i\}$ has been fixed for the interval $[t_*, \vartheta]$. Also assume that the control $u[t_*[\cdot]\vartheta)$ acting on the system is a measurable bounded function. In this case, the motion $x[t_*[\cdot]\vartheta]$ is generated as follows. Up to the instant $t_j \in \Delta_v\{t_i\}$ (which will be specified later), the part $x[t_*[\cdot]t_j]$ of this motion is

*Prikl.Matem.Mekhan.,54,3,401-408,1990

defined as the solution of the stepwise differential equation

$$x^{\cdot}[t] = A(t)\,x[t] + B(t)\,u[t] + C(t)\,v(t_i, x[t_i], \nu[t_i], \varepsilon), \quad t_i \leqslant t < t_{i+1}, \quad i = 1, \ldots, j-1 \tag{1.3}$$

The instant t_j is determined from the condition

$$\nu[t_j] - \int_{t_j}^{t_{j+1}} \langle v(t_j, x[t_j], \nu[t_j], \varepsilon)\,\Psi(t)\,v(t_j, x[t_j], \nu[t_j], \varepsilon)\rangle\,dt < 0, \ \nu[t_j] \geqslant 0 \tag{1.4}$$

From (1.4) it follows that there exists $t^* \in [t_j, t_{j+1})$ the substitution of which for the upper integration limit in (1.4) reduces the left-hand side of the first inequality in (1.4) to zero. The motion $x[t_j[\cdot]t^*]$ is defined as the continuation of the motion $x[t_*[\cdot]t_j]$ from the condition that corresponds to Eq.(1.3) in which t_i is replaced by t_j and t_{i+1} by t^*.

For $t^* \leqslant t \leqslant \vartheta$, when the disturbance slack has been fully exhausted, we assume that the disturbance does not affect the system and the motion $x[t^*[\cdot]\vartheta]$ is determined as the solution of the differential equation

$$x^{\cdot}[t] = A(t)\,x[t] + B(t)\,u[t], \quad t^* \leqslant t \leqslant \vartheta$$

Let the functional

$$\gamma = \gamma(x[t_*[\cdot]\vartheta], u[t_*[\cdot]\vartheta]) = |x[\vartheta]| + J_u(t_*, \vartheta) \tag{1.5}$$

$$J_u(\alpha, \beta) = \int_{\alpha}^{\beta} \langle u[t]\cdot\Phi(t)\,u[t]\rangle\,dt$$

be given. Here $|x|$ is some norm of the vector x that for $x \in R^n$ satisfies the condition $|x| \leqslant d\,|x|_e, d > 0$, where the symbol $|\cdot|_e$ here and in what follows stands for the Euclidean norm; $\langle u\cdot\Phi(t)\,u\rangle$ is a positive definite quadratic form, and $\Phi(t)$ is a continuous matrix function.

Let Δ_δ, $\delta > 0$, be a partition $\Delta\{t_i\}$ such that $t_{i+1} - t_i \leqslant \delta$, $i = 1, \ldots, k$. For the strategy $u(\cdot)$ and the initial position $\{t_*, x_*, \nu[t_*]\}$, the guaranteed outcome is defined as

$$c(u(\cdot), t_*, x_*, \nu[t_*]) = \overline{\lim_{\varepsilon\to 0}}\ \lim_{\delta\to 0}\ \sup_{\Delta_\delta}\ \sup_{v[t_*[\cdot]\vartheta)} \gamma$$

and for the counterstrategy $v(\cdot)$ and the initial position $\{t_*, x_*, \nu[t_*]\}$, the guaranteed outcome is defined as

$$c(v(\cdot), t_*, x_*, \nu[t_*]) = \underline{\lim_{\varepsilon\to 0}}\ \lim_{\delta\to 0}\ \inf_{\Delta_\delta}\ \inf_{u[t_*[\cdot]\vartheta)} \gamma$$

Here the lower limit is over all measurable bounded samples $u[t_*[\cdot]\vartheta]$.

The strategy $u^\circ(\cdot)$ is called optimal if $c(u^\circ(\cdot), t_*, x_*, \nu[t_*]) = \min_{u(\cdot)} c(u(\cdot), t_*, x_*, \nu[t_*])$ for any initial position $\{t_*, x_*, \nu[t_*]\}$, and the quantity $c^\circ(t_*, x_*, \nu[t_*]) = c(u^\circ(\cdot), t_*, x_*, \nu[t_*])$ is called an optimal guaranteed outcome for the position $\{t_*, x_*, \nu[t_*]\}$. We have the following proposition.

Proposition 1.1. For any initial position $\{t_*, x_*, \nu[t_*]\}$ and any counterstrategy $v(\cdot)$, we have the inequality

$$c(v(\cdot), t_*, x_*, \nu[t_*]) \leqslant c^\circ(t_*, x_*, \nu[t_*])$$

In this paper, we establish the existence of an optimal strategy $u^\circ(\cdot)$ and a counterstrategy $v^\circ(\cdot)$ that satisfies the equality $c(v^\circ(\cdot), t_*, x_*, \nu[t_*]) = c^\circ(t_*, x_*, \nu[t_*])$ for any position $\{t_*, x_*, \nu[t_*]\}$. By Proposition 1.1 this means that our game-control problem has the value $c^\circ(t_*, x_*, \nu[t_*])$ and the saddle point $\{u^\circ(\cdot), v^\circ(\cdot)\}$.

A distinctive feature of our problem is that it allows values of $u[t]\,v[t]$ as large as desired and the disturbance is bounded by integral constraints.

2. Consider the model

$$w^{\cdot} = A(\tau)\,w + B(\tau)\,u + C(\tau)\,v \tag{2.1}$$

$$w_{n+1}^{\cdot} = \langle u\cdot\Phi(\tau)\,u\rangle, \quad u \in R^r, \quad v \in R^s$$

Putting $\{w_1, \ldots, w_n, w_{n+1}\} = z \in R^{n+1}$, we rewrite (2.1) in the form

$$z^{\cdot} = A_0(\tau)z + f_0(\tau, u, v), \ u \in R^r, \ v \in R^s \quad (2.2)$$
$$f_0(\tau, u, v) = \{B(\tau)u + C(\tau)v, \ \langle u \cdot \Phi(\tau)u \rangle\}$$

where $A_0(\tau)$ is a $(n+1) \times (n+1)$ matrix. On the motions $z[\tau_*[\cdot]\vartheta] = \{w[\tau_*[\cdot]\vartheta], w_{n+1}[\tau_*, [\cdot]\vartheta]\}$ of the model (2.2) generated by the samples $u[\tau_*[\cdot]\vartheta)$, $v[\tau_*[\cdot]\vartheta)$, consider the functional

$$\gamma_0 = \gamma_0(z[\tau_*[\cdot]\vartheta], \ u[\tau_*[\cdot]\vartheta)) = |w[\vartheta]| + w_{n+1,*} + J_u(\tau_*, \vartheta),$$
$$\tau_* \in [t_0, \vartheta), \ w_{n+1,*} = w_{n+1}[\tau_*]$$

which corresponds to the functional γ (1.5). The realizations $v[\tau_*[\cdot]\vartheta)$ satisfy the constraint (1.2) with $t_* = \tau_*$.

Fix some $q > 0$ and temporarily impose an additional constraint on the sample $v[\tau_*[\cdot]\vartheta)$:

$$|v[\tau]|_e \leqslant q, \ \tau_* \leqslant \tau < \vartheta \quad (2.3)$$

Assume that some position $\{\tau_*, z_*, \nu[\tau_*]\}$ has been realized. Take a number β. The rule that associates with each piecewise-constant realization $u[\tau_*[\cdot]\vartheta)$ a piecewise-constant realization $v[\tau_*[\cdot]\vartheta)$ which satisfies (1.2) for $t_* = \tau_*$ and also (2.3) will be called a $(\beta, \nu[\tau_*]) - Q$ -procedure if $v[\tau_*[\cdot]\vartheta)$ is not anticipated by $u[\tau_*[\cdot]\vartheta)$ /1, p.223/ and any motion $z[\tau_*[\cdot]\vartheta]$ generated by this rule satisfies the inequality

$$\gamma_0(z[\tau_*[\cdot]\vartheta], \ u[\tau_*[\cdot]\vartheta)) > \beta$$

Take an arbitrary position $\{\tau, z, \nu\}$ and introduce the quantity ($\exists$ is the existential quantifier)

$$\rho^{(q)}(\tau, z, \nu) = \sup \beta, \ \beta \in B_{\{\tau, z, \nu\}} = [\beta: \exists (\beta, \nu) - Q \quad \text{procedure}] \quad (2.4)$$

The function $\rho^{(q)}(\cdot)$ has the following properties.

1°. For any numbers ν_1, ν_2 such that

$$0 \leqslant \nu_1 \leqslant \nu[t_0], \ 0 \leqslant \nu_2 \leqslant \nu[t_0], \ \nu_2 \geqslant \nu_1$$

we have the inequalities

$$\rho^{(q)}(\tau, z, \nu_2) \geqslant \rho^{(q)}(\tau, z, \nu_1), \ \rho^{(q)}(\tau, z, \nu_2) - \rho^{(q)}(\tau, z, \nu_1) \leqslant A^* \sqrt{\nu_2 - \nu_1},$$
$$\tau \in [t_0, \vartheta], \ z \in R^{n+1}$$

where A^* is a positive constant which depends on the form of the matrix functions $A(t)$, $B(t)$, $C(t)$ and $\Psi(t)$.

2°. Continuity in t and Lipschitz condition in the variable z:

$$|\rho^{(q)}(\tau, z_2, \nu) - \rho^{(q)}(\tau, z_1, \nu)| \leqslant \lambda \, |z_2 - z_1|_e$$
$$z_1, z_2 \in R^{n+1}, \ \tau \in [t_0, \vartheta], \ 0 \leqslant \nu \leqslant \nu[t_0]$$

where the constant λ is given by the equality

$$\lambda = d\sqrt{2}\,(1 + \max_{t_* \leqslant t, \, \tau \leqslant \vartheta} \|X(t, \tau)\|)$$

Here $X(t, \tau)$ is the fundamental matrix for the equation $dx/dt = A(t)x$, $\|X(t, \tau)\| = \max_y |X(t, \tau)y|_e$, $y \in R^n$, $|y|_e \leqslant 1$.

3°. $\rho^{(q)}(\tau, z, \nu) = \rho^{(q)}(\tau, \{v, 0\}, \nu) + w_{n+1}$.

4°. $\rho^{(q)}(\vartheta, z, \nu) = |w| + w_{n+1}$, where $|\cdot|$ is the same norm as in (1.5).

5°. *u-Stability property.* For any position $\{\tau_*, z[\tau_*], \nu[\tau_*]\}$, any number $\varepsilon > 0$, $\tau^* \in (\tau_*, \vartheta]$, and any piecewise-constant function $v_*[\tau_*[\cdot]\tau^*]$ satisfying the condition

$$|v_*[\tau]|_e \leqslant q, \ \tau_* \leqslant \tau < \tau^*, \ I_{v_*}(\tau_*, \tau^*) \leqslant \nu[\tau_*] \quad (2.5)$$

there exists a piecewise-constant realization $u_*[\tau_*[\cdot]\tau^*]$ such that the corresponding motion $z[\tau_*[\cdot]\tau^*]$ satisfies the inequality

$$\rho^{(q)}(\tau^*, z[\tau^*], \nu[\tau^*]) \leqslant \rho^{(q)}(\tau_*, z[\tau_*], \nu[\tau_*]) + \varepsilon(\tau^* - \tau_*), \quad (2.6)$$
$$\nu[\tau^*] = \nu[\tau_*] - I_{v_*}(\tau_*, \tau^*)$$

6°. *v-Stability property.* For any position $\{\tau_*, z[\tau_*], \nu[\tau_*]\}$, any number $\varepsilon > 0$, $\tau^* \in$

$(\tau_*, \vartheta]$, and any piecewise-constant function $u_*[\tau_*[\cdot]\tau^*]$, there exists a piecewise-constant function $v_*[\tau_*[\cdot]\tau^*]$ satisfying the condition (2.5) so that the corresponding motion $z[\tau_*[\cdot]\tau^*]$ satisfies the inequality (2.6) with $\leqslant$ replaced by $\geqslant$ and ε replaced by $-\varepsilon$.

It can be shown that the limit

$$\rho(\tau, z, \nu) = \lim_{q \to +\infty} \rho^{(q)}(\tau, z, \nu) \tag{2.7}$$

exists for any position $\{\tau, z, \nu\}$.

The function $\rho(\cdot)$, like the function $\rho^{(q)}(\cdot)$, has properties 1°-4°. Arguing along the lines of /2/ allowing for equality (2.7) and properties 2°-6° for the function $\rho^{(q)}(\cdot)$, we can establish the following result.

Theorem 2.1. For any initial position $\{t_*, x_*, \nu[t_*]\}$ of system (1.1), we have the equality $c^\circ(t_*, x_*, \nu[t_*]) = \rho(t_*, \{x_*, 0\}, \nu[t_*])$.

The optimal strategy $u^\circ(\cdot)$ is constructed as a function of the variables $\{t, x, \nu, \varepsilon\}$ in accordance with the condition

$$\langle l^\circ \cdot B(t) u^\circ \rangle + \overset{\circ}{l}_{n+1} \langle u^\circ \cdot \Phi(t) u^\circ \rangle = \min_{u \in R^r} \{\mathrm{Idem}(u^\circ \to u)\} \tag{2.8}$$

$$\rho(t, \{x - l^\circ, 0\}, \nu) - \overset{\circ}{l}_{n+1} = \min_{\{l, l_{n+1}\}} [\rho(t, \{x - l, 0\}, \nu) - l_{n+1}]$$

$$|l|_e^2 + l_{n+1}^2 \leqslant (\varepsilon + \varepsilon(t - t_0)) \exp(2\lambda(t - t_0))$$

where Idem stands for the expression on the left-hand side of the equality with the change of symbols specified in parentheses. For all $t \in [t_0, \vartheta]$, $x \in R^n$, $0 \leqslant \nu \leqslant \nu[t_0]$, $\varepsilon > 0$ we have the inequality (T denotes the transpose)

$$|u^\circ|_e \leqslant M, \quad M = \max_{t_0 \leqslant t \leqslant \vartheta} \max_{|y|_e \leqslant 1} \{|\Phi^{-1}(t) B^T(t) y|_e \lambda\} \tag{2.9}$$

3. Let us now construct a counterstrategy that generates the worst-case realizations of the disturbances. We have the following lemma.

Lemma 3.1. For any position $\{\tau, z, \nu\}$ of model (2.2), we have the inequality

$$\rho^{(a)}(\tau, z, \nu) \geqslant \rho^{(b)}(\tau, z, \nu), \quad a \geqslant b$$

Depending on the form of the matrix function $C(t)$, $t_0 \leqslant t \leqslant \vartheta$, in (1.1) and (2.1), we distinguish between non-degenerate and degenerate cases.

1. If for any $\alpha \geqslant 0$ we have the inequality

$$\max_t \|C(t)\| > 0, \quad \|C(t)\| = \max_{y \in R^s,\ |y|_e \leqslant 1} |C(t) y|_e, \quad t \in [\vartheta - \alpha, \vartheta]$$

then this is a non-degenerate case. Let us consider it in more detail.

We can prove the following proposition.

Lemma 3.2. Assume that the sphere $D_\mu = \{|x|_e < \mu,\ x \in R^n\}$ and the interval $[t_0, \alpha] \subset [t_0, \vartheta]$, $\alpha < \vartheta$ have been chosen arbitrarily. Then there is a number $q^\circ = q^\circ(\mu, \alpha)$ such that for all $q \geqslant q^\circ$ we have the limit

$$\rho^{(q)}(t, \{x, 0\}, \nu_2) - \rho^{(q)}(t, \{x, 0\}, \nu_1) \geqslant F_{\alpha\mu}(\nu_2 - \nu_1)$$

$$F_{\alpha\mu} > 0, \ \nu_2 \geqslant \nu_1, \ \nu_1 \in [0, \nu[t_0]], \ \nu_2 \in [0, \nu[t_0]], \ x \in D_\mu, \ t \in [t_0, \alpha]$$

Here $F_{\alpha\mu} \to 0$ as $\alpha \to \vartheta$.

Using the definition of $(\beta, \nu) - Q$-procedures for the position $\{\tau, z, \nu\}$ of the model (2.2) and properties 1°, 2° of the functions $\rho^{(q)}(\cdot)$, we can prove the following lemma.

Lemma 3.3. Assume that a certain number $\alpha \in [t_0, \vartheta)$ has been chosen. Then the sequence of functions $\rho^{(q)}(t, \{x, 0\}, \nu)$ uniformly converges to the optimal guaranteed outcome function $c^\circ(t, x, \nu)$ on the set $\{[t_0, \alpha] \times R^n \times [0, \nu[t_0]]\}$ as $q \to +\infty$.

Take some number $\alpha > 0$ and construct the counterstrategies $v_\alpha^{(q)}(\cdot)$ according to the condition

$$\langle l^\circ \cdot C(t) v_\alpha^{(q)}(t, x, \nu, \varepsilon) \rangle - \overset{\circ}{l}_{n+1} \langle v_\alpha^{(q)}(t, x, \nu, \varepsilon) \Psi(t) v_\alpha^{(q)}(t, x, \nu, \varepsilon) \rangle =$$
$$\min_{v \in R^s} \{\mathrm{Idem}(v_\alpha^{(q)}(t, x, \nu, \varepsilon) \to v)\}$$

$$\rho^{(q)}(t, \{x - l^\circ, 0\}, \nu - \overset{\circ}{l}_{n+1}) = \max_{\{l, l_{n+1}\}} [\rho^{(q)}(t, \{x - l, 0\}, \nu - l_{n+1})]$$

$$|l|_e^2 + l_{n+1}^2 \leqslant (\varepsilon + \varepsilon(t - t_0)) \exp(2\lambda(t - t_0)), \ t \in [t_0, \vartheta - \alpha] \tag{3.1}$$

$$v_\alpha^{(q)}(t, x, \nu, \varepsilon) \equiv 0, \ t \in (\vartheta - \alpha, \vartheta] \tag{3.2}$$

If we argue as in /1, 2/ using Lemmas 3.2, 3.3 and property 6° for the functions $\rho^{(q)}(\cdot)$, then we can prove the inequality

$$c(v_\alpha^{(q)}(\cdot), t_*, x_*, \nu[t_*]) \geqslant c^0(t_*, x_*, \nu[t_*]) - [\psi(q) + A_* \sqrt{\vartheta - \alpha}], \tag{3.3}$$
$$A_* > 0, \ \psi(q) > 0$$

for any position $\{t_*, x_*, \nu[t_*]\}$, $t_* \in [t_0, \vartheta]$, $x_* \in R^n$, $\nu[t_*] \in [0, \nu[t_0]]$; in (3.3) the function $\psi(\cdot)$ satisfies the condition $\lim \psi(q) = 0$ as $q \to +\infty$. The bound (3.3) proves the following proposition.

Theorem 3.1. For any $\zeta > 0$ there exists a counterstrategy $v_\zeta(\cdot)$ that guarantees the outcome

$$c(v_\zeta(\cdot), t_*, x_*, \nu[t_*]) \geqslant c^\circ(t_*, x_*, \nu[t_*]) - \zeta$$

for any initial position $\{t_*, x_*, \nu[t_*]\}$.

Using Lemma 3.1, Theorem 3.1, and relationship (3.3), we can establish that the counterstrategy $v^\circ(\cdot)$ that guarantees the outcome $c^\circ(t, x, \nu)$ for any initial position $\{t, x, \nu\}$ of system (1.1) is constructed according to the condition

$$\langle l^\circ \cdot C(t) v^\circ \rangle - \overset{\circ}{l}_{n+1} \langle v^\circ \cdot \Psi(t) v^\circ \rangle = \min_{v \in R^s} \{\text{Idem}(v^\circ \to v)\}$$

$$\rho^{(q(\varepsilon))}(t, \{x - l^\circ, 0\}, \nu - \overset{\circ}{l}_{n+1}) = \max_{\{l, l_{n+1}\}} [\rho^{(q(\varepsilon))}(t, \{x - l, 0\}, \nu - l_{n+1})$$

$$|l|_e^2 + l_{n+1}^2 \leqslant (\varepsilon + \varepsilon(t - t_0)) \exp(2\lambda(t - t_0)), \quad t \in [t_0, \vartheta - \alpha(\varepsilon)] \tag{3.4}$$

$$v^\circ = 0, \quad t \in (\vartheta - \alpha(\varepsilon), \vartheta] \tag{3.5}$$

where $\alpha(\cdot), q(\cdot)$ are any functions that satisfy the conditions $\alpha(\varepsilon) > 0$, $q(\varepsilon) > 0$, $\lim \alpha(\varepsilon) = 0$, $\lim q(\varepsilon) = +\infty$ as $\varepsilon \to 0$.

2. If the matrix function $C(t)$, $t_0 \leqslant t \leqslant \vartheta$ does not satisfy the conditions of the previous case, then

$$\eta^* = \min [\eta: \max_t \| C(t) \| = 0, \ \eta \leqslant t \leqslant \vartheta, \ \eta \geqslant t_0]$$

exists.

This is the degenerate case. The effect of the disturbance on system (1.1) is restricted to the time interval $[t_0, \eta^*]$. Lemmas 3.2, 3.3 and Theorem 3.1 remain basically unchanged and the counterstrategies $v_\alpha^{(q)}(\cdot)$, $v^\circ(\cdot)$ are constructed as in the first case. The only difference is that the finite time ϑ in Lemma 3.2, 3.3 and in (3.1)-(3.5) is now replaced by η^*. Therefore, in this case, we construct a counterstrategy that guarantees the outcome $c^\circ(t, x, \nu)$ for any position $\{t, x, \nu\}$ of system (1.1).

4. Let us consider a typical example. Suppose that the controlled system is described by a system of two scalar differential equations

$$x_i^{\cdot} = k u_i + n e^{\alpha t} v_i, \ t_0 \leqslant t \leqslant \vartheta, \ i = 1, 2 \tag{4.1}$$

where k, n, α are positive constants, $x = \{x_1, x_2\}$ is the phase vector, $u = \{u_1, u_2\}$ is the control vector, and $v = \{v_1, v_2\}$ is the disturbance vectors; t_0 and ϑ are fixed. Realizations of the control $u[t_0[\cdot]\vartheta]$ may be arbitrary bounded measurable functions; each realization of the disturbance $v[t_0[\cdot]\vartheta]$ satisfies the constraint

$$\int_{t_0}^{\vartheta} |v[t]|_e^2 dt \leqslant \nu[t_0], \quad \nu[t_0] > 0, \quad |v[t]|_e = (v_1^2[t] + v_2^2[t])^{1/2}$$

Assume that the initial position $\{\tau_*, x_*, \nu[\tau_*]\}$ has been chosen and the performance functional is given in the form

$$\gamma = |x[\vartheta]|_e + \int_{\tau_*}^{\vartheta} |u[\tau]|_e^2 d\tau$$

Using the results of /3/ we can establish that $c^\circ(\tau_*, x_*, \nu[\tau_*])$ is determined by the equality

$$c^\circ(\tau_*, x_*, \nu[\tau_*]) = \max_{|m|_e \leqslant 1} \{\langle m \cdot x_* \rangle + \varphi(m, \nu[\tau_*])\} \tag{4.2}$$

The function $\varphi(m, \nu)$ is determined for each m, ν $(|m|_e \leqslant 1,\ 0 \leqslant \nu \leqslant \nu[t_0])$ from the condition

$$\varphi(m, \nu) = \sup_{Q[\cdot]} \int_{\tau_*}^{\vartheta} \{\psi(m, Q[\tau], \tau)\}_* \, d\tau$$

$$\psi(m, Q[\tau], \tau) = \min_{u \in R^2} \max_{|v|_e \leqslant Q[\tau]} [k\langle m \cdot u\rangle + ne^{\alpha\tau}\langle m \cdot v\rangle + |u|_e^2], \quad \tau_* \leqslant \tau < \vartheta \tag{4.3}$$

where $\varphi(\cdot) = \{\psi(\cdot)\}_*$ denotes the upper concave hull of the function $\psi(m), |m|_e \leqslant 1$. Here the supremum is over all measurable functions $Q[\cdot]$ such that

$$\int_{\tau_*}^{\vartheta} Q^2[\tau]\, d\tau \leqslant \nu$$

In this example, for each m, ν there exists a function

$$Q^\circ_{m\nu}[\cdot] = \{Q^\circ_{m\nu}[\tau], \tau_* \leqslant \tau \leqslant \vartheta\} \tag{4.4}$$

such that the maximum in (4.3) is attained. Solving problem (4.3), we can show that the function $Q^\circ_{m\nu}[\cdot]$ is defined for each m, ν according to the values of

$$\tau_m^* = (2\alpha)^{-1} \ln[(k^2\lambda_m)/n^2]$$

$$\lambda_m = \{(|m|_e^2 - 1)k^2 + [(|m|_e^2 - 1)^2k^4 + 32\alpha n^2\nu\ (e^{2\alpha\vartheta} - |m|_e^2 e^{2\alpha\tau_*})]^{1/2}\}/(16\alpha\nu)$$

by the following relations.

$$Q^\circ_{m\nu}[\cdot] = A(2\lambda_m)^{-1} ne^{\alpha\tau}, \quad A = \begin{cases} |m^e|, & \tau_* \leqslant \tau < \tau_m^* \\ 1, & \tau_m^* \leqslant \tau < \vartheta \end{cases}$$

if $\tau_m^* \in [\tau_*, \vartheta]$, and $Q^\circ_{m\nu}[\tau] = (2\lambda_+)^{-1} ne^{\alpha\tau}$, if $\tau_m^* < \tau_*$ or $\tau_m^* > \vartheta$.

$$\lambda_+ = {}^1/_2 \sqrt{2}\,(n/(2\sqrt{\alpha\nu}))(e^{2\alpha\vartheta} - e^{2\alpha\tau_*})^{1/2}$$

Suppose that we have chosen the partition Δ_δ of the interval $[\tau_*, \vartheta]$. In accordance with the algorithm described above, the optimal control $u[\tau_i] = u^\circ(\tau_i, x[\tau_i], \nu[\tau_i], \varepsilon, \Delta_\delta)$ and the worst-case disturbance $v[\tau_i] = v^\circ(\tau_i, x[\tau_i], \nu[\tau_i], \varepsilon, \Delta_\delta)$ are constructed from the conditions

$$k\langle m^\circ[\tau_i] \cdot u[\tau_i]\rangle + |u[\tau_i]|_e^2 = \min_{u \in R^2}\{\mathrm{Idem}(u[\tau_i] \to u)\}$$

$$\max_{|g|_e \leqslant 1}\{\langle g \cdot (x[\tau_i] - m^\circ[\tau_i])\rangle + \varphi(g, \nu[\tau_i])\} = \min_m \{\mathrm{Idem}(m^\circ[\tau_i] \to m)\},$$

$$|m|_e^2 \leqslant \varepsilon + \varepsilon(\tau_i - t_0)$$

$$ne^{\alpha\tau_i} \langle l^\circ[\tau_i] \cdot v[\tau_i]\rangle - l_3^\circ[\tau_i]\,|v[\tau_i]|_e^2 = \min_{v \in R^2}\{\mathrm{Idem}(v[\tau_i] \to v)\}$$

$$\max_{|g|_e \leqslant 1}\{\langle g \cdot (x[\tau_i] - l^\circ[\tau_i])\rangle + \varphi(g, \nu[\tau_i] - l_3^\circ[\tau_i])\} =$$

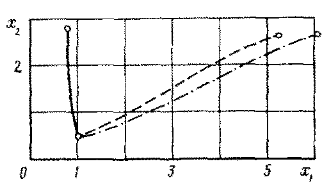
$$\max_{(l, l_3)}\{\mathrm{Idem}(l^\circ[\tau_i] \to l,\ l_3^\circ[\tau_i] \to l_3)\}, \quad |l|_e^2 + l_3^2 \leqslant \varepsilon + \varepsilon(\tau_i - t_0)$$

If at some instant τ° the disturbance slack has been completely exhausted, then we take $v[\tau] = 0,\ \tau^\circ \leqslant \tau < \vartheta$.

The control process for the given object was simulated on a computer for the following parameter values:

$$\alpha = 2,\ k = 1,\ n = 2,\ t_0 = 0,\ \vartheta = {}^1/_2,\ \tau_* = t_0$$

$$x_1[\tau_*] = 1,\ x_2[\tau_*] = {}^1/_2,\ \nu[\tau_*] = 4,\ \varepsilon = 0.03$$



In Fig.1, the solid curve represents the motion generated by the optimal control and the disturbance $v = \{0.5 \sin\tau\}$; the performance functional value is $\gamma = 3.02$; the optimal guaranteed outcome for the position $\{0, \{1, {}^1/_2\}, 4\}$ is $c^\circ = c^\circ(0, \{1, {}^1/_2\}, 4) = 6.05$, i.e., $\gamma < c^\circ$.

The dashed curve represents the motion generated by the optimal control and the worst-case disturbance; the performance functional value is $\gamma = 5.98 \approx c^\circ$.

The dash-dot curve represents the motion generated by the suboptimal control $u = \{\cos\tau, \sin\tau\}$ and the worst-case disturbance; in this case, $\gamma = 7.09 > c^\circ$.

In the last two cases, $\nu[\vartheta] \approx 0$, i.e., the initial disturbance slack for $\nu[\tau_*]$ is completely exhausted.

I would like to acknowledge the continued support of N.N. Krasovskii during this research.

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Translated by Z.L.

PMM U.S.S.R., Vol.54, No.3, pp.337-342, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

NEW SOLUTIONS OF TWO-DIMENSIONAL STATIONARY EULER EQUATIONS*

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A generalized method of separation of variables is used to obtain new particular solutions for the stream function describing two-dimensional stationary motions of an ideal fluid. Patterns of streamlines are given. The proof of the stability of some of the solutions is based on a theorem due to Arnol'd /1/.

1. In the case of the two-dimensional stationary motion of an ideal fluid, the stream function $\psi(x, y)$ satisfies the equation

$$\psi_{xx} + \psi_{yy} = \omega(\psi) \tag{1.1}$$

where ω is the vorticity. We will seek the solutions of (1.1) using the method of generalized separation of variables:

$$\psi = \alpha(f(x) + g(y)) \tag{1.2}$$

The problem arises here of finding the functions $\omega, \alpha,$ admitting of non-trivial separation of variables, i.e. a separation in which neither of the functions f, g is a polynomial of degree two or less.

Substituting expression (1.2) into (1.1), we obtain

$$(f_{xx} + g_{yy})\alpha' + (f_x^2 + g_y^2)\alpha'' = \Phi(f + g)$$

where $\Phi = \omega\cdot\alpha$. Since α' is not zero, the last equation can be written in the form

$$X + \beta Y = F(z) \tag{1.3}$$
$$X = f_{xx} + g_{yy}, \quad Y = f_x^2 + g_y^2, \quad z = f + g$$
$$\beta = \alpha''/\alpha', \quad F = \Phi/\alpha'$$

We shall call the solution of Eq.(1.3) non-trivial, if the corresponding Eq.(1.1) admits of non-trivial separation of variables.

Differentiating Eq.(1.3) with respect to x and y we obtain a relation which can be written, after dividing it by $f_x g_y$, in the form

$$2\beta'(z)X + \beta''(z)Y = F''(z) \tag{1.4}$$

Eqs.(1.3) and (1.4) can be regarded as a system of linear algebraic equations in the unknowns X, Y. The system is not inconsistent, provided that the equations are either linearly dependent, or uniquely solvable in X, Y. In the latter case we arrive at the relations

*Prikl. Matem. Mekhan., 54, 3, 409-415, 1990